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Rotation Sets of Surface Homeomorphisms

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Abstract. We consider the rotation set R of a homeomorphism f, isotopic to the identity, of a closed surface Σ of genus $g \geq 2$. We show if $\mathrm{Int}(R)$ is nonempty and contains an element which is realized by an asymptotic measure, then all the rational points of $\mathrm{Int}(R)$ are realized by periodic orbits. We raise an example to show that the second condition above is indispensable if $g \geq 2$. We also show that if R contains a $(g+1)$-simplex whose vertices are realizable by periodic orbits, then the topological entropy of f is positive.

1. Introduction

The concept of rotation vector is first introduced by M. Misiurevicz and K. Ziemian ([**9**]) for a mapping of the torus and is extended to a homeomorphism of a general manifold by M. Pollicott ([**10**]). We will give a definition applicable to an arbitrary mapping of the space, homotopic to the identity. However our definition differs from that of Pollicott in the point that we include all the vectors which correspond to invariant probabilities.

More precisely consider a continuous map f of a compact metrizable space X and a specified homotopy F joining f to the identity. Given an f-invariant probability measure μ, we shall define the rotation vector $\rho_F(\mu) \in H_1(X;\mathbb{R})$. The homology class $\rho_F(\mu)$ is said to be *realized by* the probability μ. If μ is the probability supported on a periodic orbit of f, $\rho_F(\mu)$ is said to be realized by a periodic orbit. The image of ρ_F is called the rotation set and denoted by R_F.

Although we define the rotation vector in full generality, the rest of the paper is concentrated on the case of a homeomorphism f, homotopic

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to the identity, of a closed oriented surface Σ of genus ≥ 2. In this case it is shown that ρ_F depends only on f (not on the homotopy F). Thus we denote $\rho = \rho_F$ and $R = R_F$.

The problems we treat in this paper are:

1 What is the actual figure of the set R?

2 Which points of R are realized by orbits?

3 Does R being big enough imply the complexity of f?

As for *1*, we only consider the case where f is a *generalized pseudo-Anosov homeomorphism* (GPAH). GPAH is defined following the usual definition of pseudo-Anosov homeomorphism, except we allow the (un)stable foliation to admit singularities of one-prong. There are many examples of GPAH's which are homotopic to the identity. The dynamical properties of GPAH are similar to those of pseudo-Anosov homeomorphisms. Especially they admit Markov partitions. Using this the set R is shown to be a compact polyhedron (Theorem 3).

We also prove that in the case of a GPAH, any rational point in the interior of R is realized by a periodic orbit. The proof strongly depends upon the argument of S. Alpern and V. S. Prasad [**5**].

More generally, if the limit

$$\mu(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f_*^{i-1}\delta_x$$

exists for some $x \in \Sigma$, where δ_x denotes the Dirac probability at x, the f-invariant probability $\mu(x)$ is called the *asymptotic probability* at x. Thus an asymptotic probability is closely connected with an orbit. For example an ergodic probability is asymptotic. Any periodic point admits the asymptotic probability. We show that as for a GPAH, any point in the interior of R is realized by an asymptotic probability.

For a homeomorphism of the 2–torus T^2 isotopic to the identity, J. Franks ([**4**]) showed that if R_F has a nonempty interior, then any rational point in the interior is realized by a periodic orbit. For a homeomorphism of a surface of genus $g \geq 2$, E. Hayakawa [**7**] proved that if there exists a simplex C of dimension $2g$ in R which is either disjoint from the origin or contains the origin as a vertex, and a point a in $\mathrm{Int}(C)$

such that the vertices of C, as well as a, are realized by periodic orbits, then points realizable by periodic orbits are dense in C. In this paper we generalize this result as follows. Let f be a homeomorphism of a closed oriented surface Σ of genus $g \geq 2$, isotopic to the identity.

Theorem 1. *If R contains O as an extremal point and* $\mathrm{Int}(R)$ *is nonempty and admits a point realizable by an asymptotic probability, then any rational point of* $\mathrm{Int}(R)$ *is realizable by a periodic orbit.*

By the Nielsen fixed point theorem [**8**], 0 is always realized by a fixed point.

In Theorem 1 the condition of the existence of a realizable point in $\mathrm{Int}(R)$ is indispensable if $g \geq 2$. An example will be given in Sect. 3.

As for *3*, M. Pollicott ([**10**]) showed that if there exists $2g+1$ points in R, realized by periodic points, whose convex hull has a nonempty interior, then the topological entropy of the mapping is positive. Here is an optimal result in this direction.

Theorem 2. *Let f be a C^1 diffeomorphism of Σ, isotopic to the identity. Suppose there exist points a_ν $(0 \leq \nu \leq g+1)$ in R, realized by periodic orbits, which form a nondegenerate $(g+1)$-simplex. Then the topological entropy of f is positive.*

2. Rotation sets

Let X be a compact metric space and let $F = \{f_t\}$ be a homotopy joining some mapping $f = f_1$ to the identity $id_X = f_0$. Denote by $\mathcal{M}_f(X)$ the space of f-invariant probability measures.

To any probability $\mu \in \mathcal{M}_f(X)$, let us assign a homology class $\rho_F(\mu) \in H_1(X;\mathbb{R})$, called the *rotation vector* of μ w. r. t. F. First of all, recall that

$$H_1(X;\mathbb{R}) \cong \mathrm{Hom}([X,S^1],\mathbb{R}).$$

For any $h : X \to S^1 = \mathbb{R}/\mathbb{Z}$ and $x \in X$, consider a continuous lift $h_x; [0,1] \to \mathbb{R}$ of the function $[0,1] \ni t \mapsto h(f_t(x)) \in S^1$. Denote $\Delta_F(x,h) = h_x(1) - h_x(0)$. Clearly it is independent of the choice of the lift and is continuous in x. Define $\rho_F(\mu)$ by

$$(\rho_F(\mu), [h]) = \int_X \Delta_F(x, h) d\mu(x).$$

This is independent of the choice of h from its class, as will be shown below. Suppose h is homotopic to h'. Then the function $h' - h : X \to S^1$ has a continuous lift, say $k : X \to \mathbb{R}$. Then we have

$$\int_X \Delta_F(x, h') d\mu(x) - \int_X \Delta_F(x, h) d\mu(x) = \int_X (k(f(x)) - k(x)) d\mu(x) = 0.$$

The last equality follows of course from the f-invariance of μ.

Notice that the function $\rho_F : \mathcal{M}_f(X) \to H_1(X; \mathbb{R})$ depends only on the homotopy class of F leaving the end points (the identity and f) fixed. ($\Delta_F(x, h)$ is just the same for any F in the same class.) But in general it depends on the homotopy class of F. However if we define $\widehat{\rho}_f : \mathcal{M}_f(X) \to H_1(X; \mathbb{R}/\mathbb{Z})$ in the natural way, then this depends only on f, For, if F and F' are two homotopies joining f to the identity, then the difference $\Delta_{F'}(x, h) - \Delta_F(x, h)$ is an integer. Since it is continuous, it is constant on X. Thus $\rho_{F'}(\mu) - \rho_F(\mu)$ takes an integral value on each element of $[S^1, X]$, and is an integral point of $H_1(X; \mathbb{R})$.

The proof of the following proposition is left to the reader.

Proposition 2.1. *The mapping ρ_F is continuous and affine. The image R_F is convex and compact.*

In this paper the set R_F is called the *rotation set* of F. This is slightly different from the definition in [**10**].

3. Surface homeomorphisms

Let f be a homeomorphism, homotopic to the identity, of an oriented compact surface Σ of genus $g \geq 2$.

Proposition 3.1. *The rotation vector $\rho_F(\mu) \in H_1(\Sigma, \mathbb{R})$ ($\mu \in \mathcal{M}_f(\Sigma)$) does not depend on the choice of the homotopy F joining f to the identity.*

Proof. According to Epstein ([**2**]), two homotopic homeomorphisms of the surface Σ are isotopic by an isotopy in the right class. Therefore there is an isotopy joining f and the identity in the homotopy class of

F. On the other hand, it is known by Hamstrom ([6]) that the identity component of the group of homeomorphisms of Σ is contractible. (See Earle–Eells [1] for analogous results for diffeomorphisms.) In particular any two isotopies joining f and the identity are homotopic. The proposition follows from this. $\square$

Henceforth we will denote $\rho = \rho_F$ and $R = R_F$. If there is need to indicate the homeomorphism f, we denote $\rho = \rho_f$ and $R = R_f$.

The following proposition shows the necessity of the extra condition in Theorem 1.

Proposition 3.2. *There exists a homeomorphism of the surface of genus two, isotopic to the identity, for which* $\mathrm{Int}(R)$ *is nonempty and contains no rotation vectors of asymptotic probabilities.*

Proof. Let n be a simple closed curve which cuts the surface Σ of genus 2 into two copies of punctured torus Σ_1 and Σ_2. Let l_i and m_i be the meridian and the longitude of Σ_i and let L_i and M_i be their tubular neighbourhoods. They are chosen to be very thin so that they do not meet n. Define a diffeomorphism λ_i of Σ to be the identity outside L_i, and on L_i the isotopy joining $0°$ rotation on ∂L_i to $360°$ rotation on l_i . Define a diffeomorphism μ_i likewise w. r. t. M_i. Let $f = \lambda_1 \circ \lambda_2 \circ \mu_1 \circ \mu_2$.

Consider the decomposition

$$H_1(\Sigma; \mathbb{R}) \cong H_1(\Sigma_1; \mathbb{R}) \oplus H_1(\Sigma_2; \mathbb{R}).$$

Isotoping the identity to f in a natural way, one can show from the definition that $R \cap H_1(\Sigma_i; \mathbb{R})$ coincides with the quadrilateral Q_i of vertices 0, $[l_i]$, $[m_i]$ and $[l_i] + [m_i]$.

Also it is easy to show that the rotation vector of any asymptotic probability must lie in $H_1(\Sigma_1; \mathbb{R}) \cup H_1(\Sigma_2; \mathbb{R})$. On the other hand, R is the convex hull of $Q_1 \cup Q_2$. This shows the proposition. $\square$

4. Generalized pseudo-Anosov homeomorphisms

Let $f : \Sigma \to \Sigma$ be a GPAH. Just like a usual Anosov homeomorphism, f admits a Markov partition $\{R_i \mid i = 1, \dots, n\}$. By adding the vertices and the edges to one of the neighbouring rectangles in a suitable way, we may get that $\{R_i\}$ is a disjoint measurable partition of Σ. Denote $R_{ij} = R_i \cap f^{-1}(R_j)$. Given an invariant probability $\mu \in \mathcal{M}$, form an n by n matrix $A(\mu) = (a_{ij}(\mu))$ by $a_{ij}(\mu) = \mu(R_{ij})$. It is called the *transition matrix*. The entries $a_{ij} = a_{ij}(\mu)$ satisfies (1) $\sum_{ij} a_{ij} = 1$, (2) $a_{ij} \geq 0$ and (3) $\sum_i a_{ij} = \sum_k a_{jk}$.

The set of matrices $A = (a_{ij})$ which satisfy these three conditions constitutes a polyhedron in an affine subspace of the n^2-dimensional Euclidian space. Its vertices are *cycle matrices* described as follows. Given a cyclic permutation σ of distinct $|\sigma|$ letters from $1, 2, \ldots, n$, the cycle matrix $C(\sigma)$ is defined by letting the entry $c_{ij}(\sigma)$ to be $1/|\sigma|$ if i and j occur consecutively in this order in the cyclic permutation σ and zero otherwise.

A cyclic permutation σ is called *admissible* if $\mathrm{Int}(R_{ij}) \neq \varnothing$ for any i, j occurring successively in σ. As is well known, to an admissible permutation σ is associated a periodic orbit of f. It then gives birth to an invariant probability, and hence a rotation vector, denoted by $\rho(\sigma) \in H_1(\Sigma; \mathbb{R})$.

Our results are the followings.

Theorem 3. *The rotation set R of f is the convex hull of $\rho(\sigma)$'s for admissible σ.*

Proof. All that need proof is that for any $\mu \in \mathcal{M}_f(\Sigma)$, $\rho(\mu)$ is contained in the convex hull of $\rho(\sigma)$'s. Choose a test function $h : \Sigma \to S^1$. For an isotopy F joining f and the identity, define the function $\Delta_F(x, h)$ as before. Let $h_i : R_i \to \mathbb{R}$ be a lift of the restriction of h to R_i. Choose a base point $x_i \in R_i$. Define a function $k : \Sigma \to \mathbb{R}$ by setting $k(x) = h_i(x_i) - h_i(x)$ for $x \in R_i$. Define a number

$$\Delta(i, j; h) = k(f(x)) + \Delta_F(x, h) - k(x),$$

where x is an arbitrary point of R_{ij}. Clearly this is independent of the choice of x. Let us show that

$$(\rho(\mu), [h]) = \sum_{ij} \Delta(i, j; h) a_{ij}(\mu).$$

First of all we have

$$\begin{aligned}(\rho(\mu),[h]) &= \int_\Sigma \Delta_F(x,h)d\mu(x) = \sum_{ij}\int_{R_{ij}} \Delta_F(x,h)d\mu(x)\\ &= \sum_{ij}\int_{R_{ij}} (\Delta(i,j;h) - k(f(x)) + k(x))d\mu(x)\\ &= \sum_{ij}\int_{R_{ij}} \Delta(i,j;h)d\mu(x) + \int_\Sigma (k(f(x)) - k(x))d\mu(x)).\end{aligned}$$

Now since μ is f-invariant, the last term vanishes and we have the desired equality. Likewise for an admissible permutation σ, we have

$$(\rho(\sigma),[h]) = \sum_{ij}\Delta(i,j;h)c_{ij}(\sigma).$$

Now it suffices to show that $\rho(\mu)$ is contained in the convex hull of $\rho(\sigma)$'s only for an ergodic probability μ. Since the matrices $C(\sigma)$ form vertices, we have that $A(\mu) = \sum_\sigma t_\mu(\sigma)C(\sigma)$ for some coefficients $t_\mu(\sigma) \geq 0$ with $\sum_\sigma t_\mu(\sigma) = 1$. Suppose for a while that $\sum_i \mu(\partial R_i) = 0$, where ∂R_i denotes the point set topological boundary of R_i. Then clearly the coefficient $t_\mu(\sigma)$ vanishes for a non admissible permutation σ. Now we have

$$\begin{aligned}(\rho(\mu),[h]) &= \sum_{ij}\Delta(i,j;h)a_{ij}(\mu) = \sum_{ij}\Delta(i,j;h)\sum_\sigma t_\mu(\sigma)c_{ij}(\sigma)\\ &= \sum_\sigma t_\mu(\sigma)\sum_{ij}\Delta(i,j;h)c_{ij}(\sigma) = \sum_\sigma t_\mu(\sigma)(\rho(\sigma),[h]).\end{aligned}$$

Thus we have

$$\rho(\mu) = \sum_\sigma t_\mu(\sigma)\rho(\sigma),$$

where the sum is over the admissible permutations.

Now consider the case where μ is an ergodic probability such that $\sum_i \mu(\partial R_i) > 0$. Suppose $\mu(L) > 0$ for some edge L in ∂R_i. Assume also that L lies in a stable manifold. Thus for $k > 0$, $f^k(\mathrm{Int}(L))$ is either contained in $\mathrm{Int}(L)$ or disjoint from it. At first consider the case $\mu(\partial L) = 0$. Then $\mathrm{Int}(L), f(\mathrm{Int}(L)), f^2(\mathrm{Int}(L)), \ldots$ are not mutually disjoint. For, if they were disjoint, then the total mass must be infinite. Then by the property of the Markov partition, we have

$$\mathrm{Int}(L) \supset f^k(\mathrm{Int}(L)) \supset f^{2k}(\mathrm{Int}(L)) \supset \cdots .$$

for some $k > 0$. The intersection is a singleton, say p. Since $\mu(p) = \mu(L) > 0$, the point p must be a periodic point of f. Now since μ is ergodic, μ must be the probability supported on the orbit of p. Also in the case $\mu(\partial L) > 0$, we get the same conclusion. Let R_i be a rectangle containing p in its closure. Then for some $m > 0$, we have $f^m(\mathrm{Int}(R_i)) \cap \mathrm{Int}(R_i) \neq \varnothing$. ($m$ may be a multiple of the period of p.) Choose a small open set $U \subset R_i$ containing p in its closure. For $0 \leq k \leq m$, define $\nu(k)$ by $f^k(U) \cap \mathrm{Int}(R_{\nu(k)}) \neq \varnothing$. Define a_{ij} to be l/m, where l is the number of distinct k's such that $i = \nu(k)$ and $j = \nu(k+1)$. Then clearly the matrix (a_{ij}) can be represented as a linear combination of the cycle matrices for admissible permutations. Also we have

$$(\rho(\mu), [h]) = \sum_{ij} \Delta(i, j; h) a_{ij}.$$

Now by the same calculation as before, the proof is complete. □

By Theorem 3 R is a polyhedron lying on an affine subspace. By the *affine interior* of R we mean the interior in the affine subspace of the minimal dimension containing R.

Theorem 4. *Any point of the affine interior of R is realizable by an asymptotic probability. Furthermore any rational point of the interior is realizable by a periodic orbit.*

Proof. First consider the case where v is a rational point of the interior. Then we have that $v = \sum_\sigma t(\sigma)\rho(\sigma)$ for some $t(\sigma) \geq 0$ with $\sum_\sigma t(\sigma) = 1$. The expression is not unique in general. But one can choose the coefficients so that for each admissible permutation σ, $t(\sigma)$ is positive and rational. Consider the matrix $\sum_\sigma t(\sigma)C(\sigma)$. Multiply by an integer to get an integral matrix $E = (e_{ij})$. The existence of dense orbits of f and the positiveness of the coefficients $t(\sigma)$ imply that the matrix E is irreducible, i. e. some positive power of E has all the entries > 0. Also E satisfies $\sum_i e_{ij} = \sum_k e_{jk}$.

Now consider the directed multi-graph $\mathcal{E}$ associated to E. (There exist e_{ij} oriented edges from the vertex R_i to the vertex R_j.) Since E is irreducible, $\mathcal{E}$ is directly connected, i. e. given any two vertices, there exists a directed path joining the one to the other. Also $\mathcal{E}$ is Eulerian, i. e. at each vertex the number of the edges coming in is the same as that of the edges going out. Therefore there exists an Eulerian circuit, i. e. a loop which passes through each edge exactly once. It gives birth to a periodic point of f. A routine computation as in the proof of Theorem 3 shows that the corresponding probability realizes the prescribed element v.

Next consider the case where v is an irrational point in the interior. Let $v = \sum_\sigma t(\sigma)\rho(\sigma)$. Choose a positive coefficient for each admissible permutation. Approximate the vector $t = (t(\sigma))_\sigma$ by a sequence of rational positive vectors $t^{(k)}$. Associated with the vector $t^{(k)}$ one gets an Eulerian circuit γ_k. We agree that all the circuits γ_k start and end at the same vertex, say R_1. Consider an infinite sequence formed by $\gamma_1, \gamma_2, \ldots$. There exists a point x in R_1 which realizes this sequence. If the sequence are chosen appropriately, the asymptotic probability $\mu(x)$ exists and satisfies $\rho(\mu(x)) = v$ Details are left to the reader. □

5. Proof of Theorem 1

First of all, let us study an equivalent condition for a rational class C/n to be realized by a periodic orbit, where $C \in H_1(\Sigma; \mathbb{Z})$ and $n \in \mathbb{Z}$. Let $\hat{\Sigma}$ be the maximal abelian covering of Σ, that is, the covering associated with the abelinearization $\pi_1(\Sigma) \to H_1(\Sigma; \mathbb{Z})$. The homology group $H_1(\Sigma; \mathbb{Z})$ acts on $\hat{\Sigma}$ as the deck transformation.

Let $\hat{f}_t$ be the lift of the isotopy $f_t; id \simeq f$ to $\hat{\Sigma}$ such that $\hat{f}_0$ is the identity. Denote $\hat{f} = \hat{f}_1$ and call it the *canonical lift* of f. The proof of the following proposition is left to the reader.

Proposition 5.1. *The class C/n is realizable by a periodic orbit if and only if the homeomorphism $(-C) \circ \hat{f}^n; \hat{\Sigma} \to \hat{\Sigma}$ has a fixed point.* □

Suppose that there exists a class $\rho(\mu(x)) \in \mathrm{Int}(R)$, i. e. realized by an asymptotic probability. We shall prove Theorem 1 by contradiction. So assume that a rational class C/n in $\mathrm{Int}(R)$ is not realized by a periodic point. Then we have

$$\inf\{d((-C)\circ \hat{f}^n(\hat{x}),\hat{x}) \mid \hat{x}\in\hat{\Sigma}\} > 0.$$

This shows that there exists $\epsilon > 0$ such that if $\inf\{d(g(x), f(x)) \mid x \in \Sigma\} < \epsilon$ for some homeomorphism $g; \Sigma \to \Sigma$, then $(-C)\circ \hat{g}^n$ does not have a fixed point, where $\hat{g}$ is the canonical lift of g. That is, C/n is not realized by a periodic orbit of g.

Now there exist finite extremal points μ_i of the convex set $\mathcal{M}_f(\Sigma)$ such that $\rho(\mu(x))$ and C/n lie in the interior of the convex hull of $\rho(\mu_i)$. The origin is not contained in the convex hull or else is a vertex. Since the extremal point μ_i is an ergodic probability, there exists a recurrent point x_i such that the asymptotic probability $\mu(x_i)$ exists and $\mu(x_i) = \mu_i$. In fact such points are μ_i–almost everywhere. Now choose $N_i > 0$ so large that $(1/N_i)\sum_{n=0}^{N_i-1} f_*^n\delta_{x_i}$ is arbitrarily near μ_i and that $d(f^{N_i}(x_i), x_i) < \epsilon/2$. Also for the asymptotic probability $\mu(x)$, there exist $N, M > 0$ such that $(1/N)\sum_{n=0}^{N-1} f_*^{n+M}\delta_x$ is arbitrarily near $\mu(x)$ and $d(f^{N+M}(x), f^M(x)) < \epsilon/2$.

For any i, let $\delta_i : [0,1] \to \Sigma$ be the minimal geodesic such that $\delta_i(0) = f^{N_i}(x_i)$ and $\delta_i(1) = x_i$. Consider the product $\Sigma\times[0,1]$ and the curve $\hat{\delta}_i$ defined by $\hat{\delta}_i(t) = (\delta_i(t), t)$. Also let $\delta : [0,1]\to\Sigma$ be the minimal geodesic such that $\delta(0) = f^{N+M}(x_i)$ and $\delta(1) = f^M(x_i)$, and define a curve $\hat{\delta}$ by $\hat{\delta}(t) = (\delta(t), t)$. By modifying slightly the curves $\hat{\delta}_i$ and δ_i if necessary, one may assume that they are mutually disjoint and also disjoint from $f^n(x_i)\times[0,1]$ $(0 < n < N_i)$ and $f^n(x)\times[0,1]$ $(M < n < N+M)$. Choose a small tubular neighbourhood of $\hat{\delta}_i$ and $\hat{\delta}$. Define a vector field $X = (Y_t, \partial/\partial t)$ such that $X = (0, \partial/\partial t)$ outside the union of the tubular neighbourhoods and X is tangent to $\hat{\delta}_i$ and $\hat{\delta}$. Define a homeomorphism ψ of Σ by mapping the initial point of the orbit of X to its terminal point. Then ψ maps the point $f^{N_i}(x_i)$ to x_i and $f^{N+M}(x)$ to $f^M(x)$. If the modification of $\hat{\gamma}_i$ and $\hat{\gamma}$ is small and if the flow X is chosen appropriately, then one has $d(x,\psi(x)) < \epsilon$.

Let $g = \psi f$. Then the points x_i and $f^N(x)$ becomes periodic points of g. Since

$$(1/N_i)\sum_{n=0}^{N_i-1} f_*^n\delta_{x_i}$$

is arbitrarily near μ_i and

$$(1/N)\sum_{n=0}^{N-1} f_*^{n+M}\delta_x$$

is arbitrarily near $\mu(x)$, one may assume that $\rho_g(\mu(x))$ and C/n also lie in the interior of the convex hull of $\rho(\mu(x_i))$, where $\mu(x)$ e. t. c. is of course w. r. t. the homeomorphism g. If some vertex is the origin, we make no modification for it.

From now on, we only consider g. First of all notice that the class C/n is not realized by a periodic point. The points x_i as well as $y = f^M(x)$ are periodic points of g. $\rho(\mu(y))$ and C/n lie in the interior of the convex hull of $\rho(\mu(x_i))$. The relation of the origin and the convex hull is unchanged.

Now let $\mathcal{O}$ be the union of orbits of x_i. Assume for contradiction that there exists a simple closed curve γ in $\Sigma\setminus\mathcal{O}$ which is essential in Σ, such that $g(\gamma)$ is isotopic to γ by an isotopy which keeps $\mathcal{O}$ invariant. If γ is nonseparating, then it is easy to show that for any asymptotic probability $\mu(x)$, the intersection number of the homology class $[\gamma]$ of γ and $\rho(\mu(x))$ vanishes. This is contrary to the fact that the convex hull of $\rho(\mu(x_i))$ has nonempty interior.

On the other hand if γ separates Σ into subsurfaces Σ_1 and Σ_2, then in the decomposition

$$H_1(\Sigma;\mathbb{R}) = H_1(\Sigma_1;\mathbb{R})\oplus H_1(\Sigma_2;\mathbb{R}),$$

the rotation vector $\rho(\mu(x))$ of any asymptotic probability lies either on $H_1(\Sigma_1;\mathbb{R})$ or on $H_1(\Sigma_2;\mathbb{R})$. This contradicts the fact that $\rho(\mu(y))$ lies in the interior of the convex hull $\rho(\mu(x_i))$. Recall the position of the origin.

But there may exist a simple closed curve γ, essential in $\Sigma\setminus\mathcal{O}$ but trivial in Σ such that $g(\gamma)$ is isotopic to γ relative to $\mathcal{O}$.

If there does not exist such a curve, then g is isotopic to a GPAH g' by an isotopy keeping $\mathcal{O}$ fixed. Then by the result of the previous section, the class C/n is realizable by a periodic orbit $O(z)$ of g'. Now $O(z)$ is homotoped to a periodic orbit of g which also realizes C/n. A contradiction.

In case there are curves γ_i $(1 \leq i \leq q)$ mentioned above, one may suppose that they are disjoint and bound discs D_i in Σ. Then g is isotoped to g' relative to $\mathcal{O}$ which keeps $\cup_i \gamma_i$ invariant and is a GPAH on $\Sigma \setminus \cup_i D_i$. By an analogous argument we get also a contradiction.

6. Proof of Theorem 2

The rotation set (resp. topological entropy) of the k times iterate of f is the k times scalar product of the rotation set (resp. topological entropy) of f. Therefore we are free to pass to an iterate of f in the proof. Now suppose on the contrary that the topological entropy of f is zero. Passing to an iterate if necessary, one may assume that any a_ν in Theorem 2 corresponds to a fixed point x_ν of f. Blowing up f at all the points x_ν's, we get a diffeomorphism g of a surface Σ_0 with boundaries corresponding to x_ν's. Notice that the topological entropy of g is still zero. According to W. Thurston, there exists a system $\sigma = \{s_i\}$ of disjoint simple closed curves of Σ_0, and a homeomorphism g' isotopic to g which keeps a tubular neighbourhood N of σ invariant, and on any component Σ_j of $\Sigma_0 \setminus N$, if $(g')^{n_j}(\Sigma_j) = \Sigma_j$ for some $n_j > 0$, then $(g')^{n_j}|_{\Sigma_j}$ is either periodic or pseudo Anosov.

Passing to an iterate if necessary, one may assume that all the subsurfaces Σ_j are kept fixed by g'. Now $g'|_{\Sigma_j}$ must be periodic. For if not, the topological entropy of g', and hence by Thurston ([**3**]), of g must be positive. Again passing to an iterate, one may assume that $g'|S_j$ is the identity on any Σ_j. That is, g' is a composite of Dehn twists along the curves s_i's.

Let γ be a path in Σ joining x_ν to x_μ. Then clearly we have

$$[f(\gamma)] - [\gamma] = [g'(\gamma)] - [\gamma] = a_\mu - a_\nu.$$

Therefore $a_\mu - a_\nu$ is contained in the subspace V of $H_1(\Sigma;\mathbb{R})$ spanned by $[s_i]$'s. Since s_i's are mutually disjoint, we have $\dim V \leq g$. A contradiction.

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